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Orbit Determination by Solving for Gravity Parameters with Multiple Arc Data

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The orbit of a satellite that repeats in the Earth fixed coordinates is determined by combining GPS tracking data from multiple arcs. The satellite dynamics are modeled with the epoch state and a set of parameters, called the bin parameters, that account for the effect of the local gravitational field on the satellite current state. The epoch state is specific to each arc, and the bin parameters are common to all repeat arcs. The estimation algorithm is based on the Square Root Information Filter. It involves partitioning of the measurement matrix and use of the Householder transformation to combine multiple arc data and solve for the epoch states and the bin parameters. The bin parameters can then be converted into the Earth's gravitational field with a modest amount of computation.

Introduction

A MODEL of the Earth's gravitational field is commonly used in satellite orbit determination. In the conventional dynamic tracking technique, models of the Earth's gravitational field and other forces are required to model the dynamics of the spacecraft; the model orbit is then adjusted to fit the tracking data. As a result, modeling error in the gravitational field leads to error in the orbit solution. For high accuracy applications at low to midaltitude, gravitational mismodeling can be the dominant error.[1-4]

Modeling the dynamics to determine the orbit is necessary for a tracking system that lacks continuous coverage or precise measurements. This is the case with ground-based Doppler or laser tracking systems due to their limited coverage. The Global Positioning System (GPS) will provide continuous three-dimensional coverage and allows alternative approaches. The nondynamic, or kinematic, technique[5] performs point positioning by using the pseudorange and carrier phase data and is not dependent on a gravitational model. The reduced-dynamic technique[6-8] uses dynamic models, but its dependence on the dynamics is relaxed. The information in the dynamic model and the geometric strength of the measurements are combined in an optimal way. The solution is therefore less sensitive to gravitational and other dynamic mismodeling.

Gravitational models are traditionally constructed from satellite tracking data or surface gravimetric measurements or a combination of both. The gravitational potential is most often expressed as a spherical harmonic expansion.[9,10] To construct the gravitational model, the coefficients for the spherical harmonic functions are also adjusted to fit the tracking data. A good model requires a large number of coefficients and the processing of a large data set from multiple satellites. The basic technique is much the same as that used for dynamic tracking except that the gravitational coefficients are also adjusted. Because of the large number of estimated parameters and the volume of data processed, this is a computationally intensive endeavor.

For some satellites with special orbits, the computation can be reduced with refined methods. Colombo[11-13] has exploited the symmetry properties of a repeat orbit and shown that a sparse block-diagonal and bordered normal matrix can be set up and inverted efficiently. This paper describes another technique[14,15] for orbit determination and gravitational field recovery with a repeat orbit that requires less computation. We assume the satellite repeats its ground tracks after several days and has either frozen or nearly circular orbit so that the orbit in the Earth fixed coordinates is periodic. As the satellite repeats its ground track, it experiences the same gravitational perturbation from the Earth. The perturbation for arcs with the same ground track can be modeled by a set of parameters that represent the effect of the local gravitational field. Tracking data from multiple repeat arcs can be processed together to obtain an accurate estimate of the parameters. The choice of local gravitational parameters for modeling the dynamics allows the partitioning of the measurement matrix in the regression equation and the use of an orthogonal transformation to process multiple arc data for recovery of spherical harmonic coefficients. This approach is more computationally efficient than the conventional method of solving for the gravitational field because of the sparseness of the matrix involved.

This technique is suited to a number of satellites being planned, such as Topex/Poseidon,[16,17] EOS,[18] ERS-1,[19] and Aristoteles.[20] The Topex/Poseidon satellite is set for launch in 1992 to carry out an ocean topography experiment that will measure the global sea surface using a precise radar altimeter, with the principal purpose of charting global ocean circulation. The orbit has an altitude of 1336 km, an inclination of 66.1 deg, and an eccentricity less than 0.001. The ground tracks repeat every 10 nodal days after 127 revolutions.[21] The baseline tracking system for Topex is a ground-based laser ranging system. In addition, a GPS-based tracking system is being carried as an experiment. A key mission objective is to achieve 13-cm accuracy in determining Topex geocentric altitude. Extensive orbit analysis has shown that gravitational mismodeling is a major error source using the traditional dynamic orbit determination technique, even with a much improved model.[22-25] The repeat orbit technique can be applied to the GPS tracking system to obtain a high accuracy orbit, and the orbit can be converted into the gravitational field with a modest amount of computation.

Satellite Dynamic Model

Our gravitational dynamic model is based on the assumption that the satellite repeats its orbit in an Earth fixed coordinate frame after a number of revolutions, and the perturbation caused by the error of the nominal gravitational field is the same for all repeat orbits. Each orbit may have slightly differ-

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ent initial position and velocity, but since they are close together, they experience effectively the same perturbation due to the Earth's gravitational field. Measurement data from these arcs may be collected into segments, or "bins." By averaging the measurement data from multiple arcs, the effect of random errors is reduced in the estimation of the perturbation.

Because the repeat orbits deviate from each other slightly, the perturbations are only approximately the same. However, the repeat is not required to be exact, and minor variation compared to the satellite altitude is acceptable for the method proposed here. For example, the ground track deviation for Topex/Poseidon is in the order of 1 km,[26] and the eccentricity is less than 0.001. Since the variation in ground track and altitude is very small compared with the altitude of 1336 km, the error introduced by the approximation is negligible. Over a few kilometers, the total gravitational field varies substantially. But since the nominal field is already removed in the estimation process and we are only modeling the effect due to the difference between the real gravitational field and the nominal field, the variation in distance is acceptable.

In adjusting the gravitational model with the spherical harmonic expansion, the general rule is to limit the highest degree terms to those with wavelength comparable to the satellite altitude, since the sensitivity of the tracking data to the harmonic terms decreases sharply as the degree increases. The highest degree of the adjusted terms is about 50 for Topex/Poseidon, which corresponds to a wavelength of roughly 1000 km. A variation of a few kilometers is a small fraction of this wavelength and will only introduce a small error in the perturbing gravitational field (i.e., the error of the nominal gravitational field). A somewhat different justification can be made by attributing the perturbing field to a mass concentration below the surfaces of the Earth. In the worst case, this mass concentration is directly below the satellite. The variation of the repeat orbit is two orders of magnitude smaller than the distance between the satellite and the mass concentration. The variation in the perturbing gravitational field due to the mass concentration is therefore about two orders of magnitude smaller than the perturbing gravitational field itself. It is therefore justifiable to assume that the perturbing gravitational field is the same for all repeat orbits. This also implies that the resulting perturbations of different orbits are the same.

To represent the satellite dynamic model, assume we have N arcs with epoch time $t_0^{(i)}$, where the superscript i labels the arc and $i = 1, \ldots, N$. At each epoch time, the satellite is assumed to start the arc from the same location above the beginning of the ground track. The measurement is made at $t_j^{(i)}$ with $j = 1, \ldots, M$ and $i = 1, \ldots, N$. The superscript labels the arc, and the subscript labels the measurement time. The epoch time may or may not be the same as the first measurement time in each arc. But, in general, the measurement schedules should be the same for all arcs. In other words,

$$t_j^{(i)} - t_0^{(i)} = t_j^{(i')} - t_0^{(i')} \qquad \text{for} \qquad i \neq i' \tag{1}$$

At time $t_j^{(i)}$ for the same j but different i, the satellite is approximately over the same location of the ground track.

The satellite state at any time in an arc depends on the initial state and the dynamical force for that arc. We can write

$$s\left[t_j^{(i)}\right] = \Phi\left[t_j^{(i)}, t_0^{(i)}\right] s\left[t_0^{(i)}\right] \tag{2}$$

where $s[t_0^{(i)}]$ is the epoch state for the ith arc, and $\Phi[t, t_0^{(i)}]$ is the state transition operator, which is dependent on the gravitational force and other dynamical forces such as drag and solar radiation. The gravitational force due to the Earth is assumed to be the same for all arcs whereas drag and solar radiation, as well as the gravitational forces due to the Sun and the Moon, vary for different arcs. The model trajectory is generated from a nominal epoch state using a dynamic model with gravitational and nongravitational forces. The nominal trajectory will be arc specific due to forces that are arc specific.

In discussing the measurement data type, we will limit ourselves to pseudorange and carrier phase, which is also called integrated Doppler. With these data types, the velocity of the satellite at the measurement time is not directly involved in the measurement model, although the initial velocity is. It is convenient to separate the position component of the satellite initial state from the velocity component in our discussion.

We can linearize the previous equation and get

$$r(t) = \frac{\partial r}{\partial r_0} r_0 + \frac{\partial r}{\partial v_0} v_0 + \delta(t) + \frac{\partial r}{\partial p} p \tag{3}$$

where $r(t)$ is the deviation of the actual satellite position from the nominal trajectory, r_0 is the adjustment to the epoch position, and v_0 is the adjustment to the epoch velocity. The term $\delta(t)$ represents the correction to the current position of the satellite to account for the error in the nominal gravitational model. The last term is an adjustment to all other nongravitational forces such as the drag and radiation pressure. Nongravitational dynamic parameters are estimated if the forces are not sufficiently well known for the particular application. Whether an individual force parameter needs to be adjusted or not depends on arc length and other factors. More discussion on the choice is given in the section on recovering gravitational field. The partial derivatives in the previous equation are evaluated with the nominal dynamic model.

The corresponding equation for the velocity component is

$$\dot{r}(t) = \frac{\partial \dot{r}}{\partial r} r_0 + \frac{\partial \dot{r}}{\partial v_0} v_0 + \dot{\delta}(t) + \frac{\partial \dot{r}}{\partial p} p \tag{4}$$

At measurement time $t_j^{(i)}$ we can write the equation for the position component as

$$r\left[t_j^{(i)}\right] = \frac{\partial r}{\partial r_0} r_0 + \frac{\partial r}{\partial v_0} v_0 + \delta_j + \frac{\partial r}{\partial p} p \tag{5}$$

where

$$\delta_j = \delta\left[t_j^{(i)}\right] \tag{6}$$

In the estimation process, a detailed model of the gravitational field is not needed explicitly. Instead, we have a set of $(3 \times M)$ parameters to describe the effect of gravitational force on the position component of the orbit. These parameters henceforth will be called the bin parameters.

A word on the bin parameter is in order. The parameter is essentially the second time integral of the perturbing gravitational field (i.e., the error of the nominal gravitational field). As will be shown by Eq. (19) in the section on recovering the gravitational field, it can be interpreted as the variational partial of the gravitational field. If one pictures the dynamics in terms of the spherical harmonic coefficients, then δ_j represents the lumped effect of all of the harmonic coefficients on the current state.

The correction term δ is considered common to all arcs and will be adjusted later in the estimation process. The assumption that the deviation of the current state due to gravitational model deviation is the same for all arcs is intuitively reasonable, but it should be more carefully justified as will be done later in the section on recovering the gravitational field. The adjustment terms to the epoch state, r_0 and v_0, are considered arc specific. At a selected measurement time t_1, the position and velocity of the satellite are given by

$$r(t_1) = \frac{\partial r_1}{\partial r_0} r_0 + \frac{\partial r_1}{\partial v_0} v_0 + \delta_1 + \frac{\partial r_1}{\partial p} p \tag{7}$$

and

$$\dot{r}(t_1) = \frac{\partial \dot{r}_1}{\partial r_0} r_0 + \frac{\partial \dot{r}_1}{\partial v_0} v_0 + \dot{\delta}_1 + \frac{\partial \dot{r}_1}{\partial p} p \tag{8}$$

We can impose the condition that both δ and $\dot{\delta}$ at t_1 are arc common because each arc has its own adjustment term to the epoch state. Furthermore, since the parameters r_0, v_0, and $\delta(t)$ together give more degrees of freedom for the satellite positions than necessary, we may set one of the vectors $\delta(t)$ to zero to reduce the number of estimated parameters. This is done by choosing $\delta(t)$ to be zero at t_1, that is,

$$\delta(t_1) = \delta_1 = 0 \tag{9}$$

However, we will not assume that the velocity component $\dot{\delta}_1$ is zero. This is because with the measurement data type to be considered, namely range and carrier phase, the velocity component of the satellite is not directly observable and $\dot{\delta}_1$ cannot be estimated. Appendix A discusses this ambiguity in more detail. With the assumption that the initial condition of the correction term δ is arc common, and the assumption that the gravitational fields repeat, we can show that the correction term is arc common at any time. This is done later when the variational equation is discussed.

Note that unlike the case where the dynamic model is used, here the term v_0 is not the actual initial velocity, but rather it is the adjustment to the initial velocity. The satellite initial velocity is the sum of v_0 and $\dot{\delta}_0$.

Measurement Model

The nominal GPS tracking system consists of 21 GPS satellites, the low Earth satellite equipped with a GPS receiver, and a number of ground receivers. The P-code pseudorange and carrier phase measurements are collected by both the orbiting and ground receivers. For orbit determination, each measurement arc is roughly 2 h long, which corresponds to one revolution for the low Earth satellite. Longer arcs, typically 10 days long, are often needed for gravity work. Measurements are taken at intervals of 5 min or less.

The measurement at time t is modeled by the regression equation, which has also been linearized and is shown in Eq. (10). The dynamic parameters mentioned earlier, plus nondynamic parameters used in modeling the measurement, will appear in the regression equation for the measurement. The parameter δ will appear in the regression equation as the gravitational parameters to be estimated. Note that there are three gravitational parameters for each measurement time if pseudorange or carrier phase is used, since they depend only on the position components of the gravitational parameter δ. If (instantaneous) range rate measurement is used, the velocity components of the gravitational parameter, and to a certain extent the position components, are also involved.

To simplify the discussion, we will limit the measurement model and data processing strategy to pseudorange and carrier phase only. Let ρ be the linearized pseudorange measurement (i.e., pseudorange deviation from the nominal value) between the low Earth orbiter and the GPS satellite,

$$\rho\left[t_j^{(i)}\right] = \frac{\partial \rho}{\partial r_0} r_0 + \frac{\partial \rho}{\partial v_0} v_0 + \frac{\partial \rho}{\partial \delta_j} \delta_j + \frac{\partial \rho}{\partial p} p + \frac{\partial \rho}{\partial q} q + \epsilon \tag{10}$$

where q represents all parameters not related to the dynamics of the low Earth satellite that are involved in the measurement, such as GPS state and clocks. The last term is the measurement noise.

Also involved in the orbit determination process but not written explicitly here is the equation for the pseudorange measurement between the GPS satellite and the ground station. The parameters for the measurement include the tropospheric delay, the station locations, and the station clock parameters. The troposphere and clock parameters are specific to each arc whereas the station location is common to all arcs. The measurement model for carrier phase is similar except that additional range bias parameters are included. A range bias is introduced for each session of uninterrupted phase count between a receiver and a GPS satellite. This parameter needs to be estimated and is arc specific.

We next combine terms in the previous equation to write it in the form

$$\rho\left[t_j^{(i)}\right] = \frac{\partial \rho}{\partial u^{(i)}} u^{(i)} + \frac{\partial \rho}{\partial w} w + \epsilon \tag{11}$$

where $u^{(i)}$ denotes all arc specific parameters, including the adjustment term to the epoch state r_0 and v_0, the nongravitational dynamical parameters, and GPS states. Arc common parameters such as the bin parameters are represented by w in the second term on the right. The separation of parameters into arc specific and arc common parameters is important for the estimation algorithm discussed next.

In using this measurement model and estimating the adjustment to the epoch state and the bin parameters later, we have in fact overparameterized the system. The adjustment terms for the initial state of the satellite and the gravitational parameters are dependent on each other. For multiple arcs, we can eliminate the v_0 term for one of the arcs as the extra parameter. If all parameters are kept, it is not possible to completely determine them with the measurements and numerical difficulty because singularity may be encountered in calculation. To avoid the singularity, we can use large a priori sigmas to loosely bound the solution. Alternatively, we can eliminate the extra velocity parameter in our formulation. Either way, the estimation algorithm discussed later is applicable. Physical quantities of interest to us, namely, the satellite positions at different times and the gravitational field, can always be determined unambiguously from the measurements. More details are given in Appendix A.

Estimation Algorithm

The number of arc specific parameters increases as the number of arcs increases. Using a straightforward algorithm, the amount of computation can be very large for a large number of arcs. Much saving in computation can be achieved by the algorithm described next that takes advantage of the sparseness of the measurement matrix.

If we combine the measurement data from multiple arcs, we can write the regression equation that models the measurement in matrix form

$$y = Ax + \epsilon \tag{12}$$

where y is the measurement column vector and ϵ is the measurement noise column vector. The measurement noise is assumed to be uncorrelated and have the same standard deviation, i.e., sigma. If the noise sigmas are different, i.e., the measurements have different accuracies, Eq. (12) may be scaled to make the standard deviation of the noise uniform. The column vector x denotes the parameters that have been classified into arc specific and arc common parameters. The measurement matrix A, also known as the design matrix, contains the partial derivatives and can be partitioned into an arc specific part and an arc common part. The A matrix has the form

$$A = [A_1 | A_2] \tag{13}$$

where the submatrix A_1 contains partials of arc specific parameters and is block diagonal, with each block corresponding to one arc. The submatrix A_2 contains partials of arc common parameters and is generally full. From the appearance of the A matrix, it can be seen that an algorithm using the Householder transformation[27] can be applied to solve the regression equation, in a similar manner as the linear combination method for elimination of clock parameters,[28] or the method for solving for multiple mission station locations.[29] The algorithm is based on the Square Root Information Filter that transforms the measurement matrix and data by an orthogonal transformation to obtain the solution and its covariance.

Figure 1 shows graphically the forms of the A matrix as it is transformed in a series of steps. Shaded areas represent nonzero portions, and blank areas represent zero portions of the matrix. Part a of the figure shows the original form of the A matrix. The matrix has been partitioned horizontally into three arcs and vertically into parts for arc specific on the left and arc common parameters on the right. To carry out the estimation scheme, we first partially triangularize a portion of the A matrix involving the arc specific parameters for each arc and retain the lower part of the triangularized matrix for arc common parameters. The result is shown in part b. The matrix is rearranged by combining the lower submatrices from different arcs to form a rectangular submatrix at the lower part of A as shown in part c. The combined part is further triangularized, as shown in part d. This completes the triangularization of the entire A matrix. The lower right part of the matrix is used to solve for the arc common parameters, namely, the gravitational parameters. The solution is then used together with the upper part of the triangularized matrix for each arc to solve for the corresponding arc specific parameters. This way substantial savings can be made in computation, and the result is the same as that obtained without partitioning the A matrix.

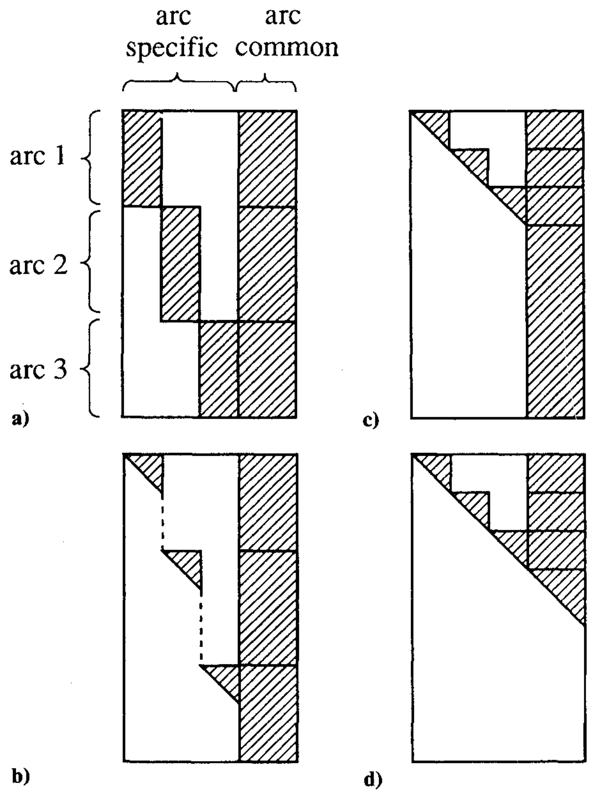

Fig. 1 Transformation of the A matrix.

Recovering Gravitational Field

As stated earlier, this method allows conversion of the orbit solution to the gravitational field with modest computation. The direct product of the estimation algorithm is the bin parameter δ, which is a correction term to the satellite current position to account for the error of the nominal gravitational field. To recover the gravitational field itself, this set of parameters must first be related to the gravitational field. There are two ways to establish the relationship: the first one is based on the equation of motion, and the second one is based on the variational equation.

In the first approach, the linearized equation of motion is written with the conventional gravitational parameters as

$$r(t) = \frac{\partial r}{\partial r_0} r_0 + \frac{\partial r}{\partial v_0} v_0 + \sum_{\mu}^{\mu_{\max}} \frac{\partial r}{\partial k_\mu} k_\mu + \frac{\partial r}{\partial p} p \tag{14}$$

where k_μ are the spherical harmonic coefficients or the mass concentration parameters of the gravitational model. Comparison with Eq. (3) shows that the bin parameter can be expressed as

$$\delta = \sum_{\mu}^{\mu_{\max}} \frac{\partial r}{\partial k_\mu} k_\mu \tag{15}$$

This provides the basis for further least squares estimation of the parameters k_μ. Equation (15) can be substituted into Eq. (12) and the transformed regression equation can be further processed to give an optimal solution of the spherical harmonic coefficients.[30] Conceptually, this is equivalent to using the gravity bin parameters as correlated pseudomeasurements to solve for the gravitational parameters. This approach gives the same result as the conventional method that estimates the coefficients directly from the measurements. The additional computation to convert the gravity bins to the spherical harmonic coefficients needs to be done only once for multiple arcs and is much less than the earlier part of the computation that estimates the bin parameters.[31]

In the second approach that will be discussed next, we consider the variation of the equation of motion for the satellite. The parameter δ can be interpreted as a variational partial that satisfies the variational equation. A physical interpretation of the variational equation will be given. A formula derived from the variational equation enables us to compute the local gravitational field.

The force field is defined as force per unit mass on the satellite. Let the actual force field be denoted by F_{real} and the nominal force field by F_n. The difference between the two is represented as the sum of two terms,

$$F_d + F_p = F_{\text{real}} - F_n \tag{16}$$

where F_d is due to the Earth's gravitational force, and F_p is due to nongravitational forces. The force field F_d is assumed to be arc common, but F_p and F_n are generally arc specific. The arc specific forces should be taken into account in generating the nominal trajectory. However, in the variational equation to be discussed, the partial derivatives of the nominal force field are considered arc common.

We next consider a variational force field defined as

$$F_{\text{variational}} = F_n + F_p + \alpha F_d \tag{17}$$

The variational parameter α is introduced with the understanding that when it is equal to 0 the variational force field reflects the nongravitational and the nominal gravitational models, and when it is equal to 1 the variational force field is equal to the actual field. The nominal force field does not need to be very accurate. It only needs to be accurate enough to permit linearization of the equation of motion.

With this assumption, the satellite position at time t is dependent on the adjustment terms to its epoch state, the coefficient α, and other dynamical parameters. If we linearized the equation by considering the variation from a nominal trajectory, we have

$$r(t) = \frac{\partial r}{\partial r_0} r_0 + \frac{\partial r}{\partial v_0} v_0 + \frac{\partial r}{\partial \alpha} \alpha + \frac{\partial r}{\partial p} p \tag{18}$$

where p represents the nongravitational dynamical parameters mentioned earlier. Within the assumption of linearity, this equation is valid for an arbitrary value of α, in particular for $\alpha = 0$ and $\alpha = 1$. When $\alpha = 0$, the equation represents the satellite position with the nominal gravitational field. When $\alpha = 1$, the equation represents the satellite position with the real gravitational field. Notice that the gravitational parameter δ defined earlier is the difference of the satellite positions with the real gravitational field and with the nominal gravitational field.

We also have the linearized equation for the satellite state given earlier as

$$r(t) = \frac{\partial r}{\partial r_0} r_0 + \frac{\partial r}{\partial v_0} v_0 + \delta(t) + \frac{\partial r}{\partial p} p \tag{19}$$

By setting α to 1 and comparing the two equations, we see that the bin parameter δ is related to the variational partial by

$$\delta = \frac{\partial r}{\partial \alpha} \tag{20}$$

The gravitational parameter δ was earlier interpreted as the correction to the satellite position at time t to account for the error in the nominal gravitational model. By the earlier argument it can also be considered a variational partial. A variational partial is related to the gravitational field by the variational equation that is derived from the equation of motion (Ref. 32, pp. 64–68).

Let r be the position of the satellite at time t. Let β be a general dynamic parameter, which can be the gravitational variational parameter α defined earlier and can also be another parameter such as the satellite initial position, initial velocity, the drag coefficient, or the solar radiation parameter. The equation of motion is given by

$$\ddot{r} = F(\dot{r}, r, \beta) \tag{21}$$

where F is the force per unit mass acting on the satellite. Let

$$\zeta(t) = \frac{\partial r}{\partial \beta} \tag{22}$$

Differentiating the equation of motion with respect to β and interchanging the order of differentiation, we get

$$\ddot{\zeta} = \frac{\partial F}{\partial \dot{r}} \dot{\zeta} + \frac{\partial F}{\partial r} \zeta + \frac{\partial F}{\partial \beta} \tag{23}$$

This is the variational equation. In an orbit determination problem, the variation equation and a certain initial condition are usually used to calculate the variational partials by integration with a force of known time dependence. Here the step is reversed. The partial derivatives in the equation cannot be computed from the gravity, since we have no accurate knowledge of the actual gravitational field. Instead, the variational equation is used to relate the partials to the gravity. With the variational partials computed by the estimation algorithm, we will infer the gravitational field from the variational partials. This is done next.

Since the gravitational parameter δ can be considered a variational partial, it satisfies the variational equation. We therefore set

$$\beta = \alpha \tag{24}$$

and

$$\zeta = \delta \tag{25}$$

Using the definition Eq. (15) of the variational gravitational field, and the nominal force field to evaluate the partial derivatives, we have

$$\ddot{\delta} = \frac{\partial F_n}{\partial \dot{r}} \dot{\delta} + \frac{\partial F_n}{\partial r} \delta + F_d \tag{26}$$

This equation has the following physical meaning: The variational partial δ is the deviation of the actual orbit from the nominal orbit due to gravitational mismodeling. Its second time derivative $\ddot{\delta}$ is the correction to the acceleration. The first and second terms on the right-hand side are the changes in the nominal force field due to the perturbation of the orbit position and velocity, respectively. The third term is the correction to the nominal force field at a fixed point in space. The right-hand side is thus the total correction to the force field on the satellite. The variational equation states that the correction to the satellite's acceleration is equal to the correction to the force field it experiences.

This equation also provides the justification that the gravitational parameter δ is arc common. Since it is a second-order differential equation, the solution of the equation is uniquely determined by the gravitational fields F_n and F_d, together with an initial condition, i.e., the value and the first time derivative of δ at a chosen time, for example t_1. Our basic assumption is that the gravitational field F_d is arc common. We also assume that both the value and the first time derivative of δ at t_1 are arc common. With these assumptions, there is a unique, i.e., arc common, solution for the equation. Therefore the gravitational parameter δ can be considered arc common.

The equation can be rewritten as

$$F_d = \ddot{\delta} - \frac{\partial F_n}{\partial \dot{r}} \dot{\delta} - \frac{\partial F_n}{\partial r} \delta \tag{27}$$

If the correction to the nominal force field F_d is reasonably smooth over twice the measurement interval, the second derivative of the variational partial can be approximated by finite differencing of the gravitational parameters,

$$F_d(t_j) = \frac{1}{\Delta t^2}(\delta_{j+1} + \delta_{j-1} - 2\delta_j) + \frac{1}{2\Delta t} \frac{\partial F_n}{\partial \dot{r}} (\delta_{j+1} - \delta_{j-1}) - \frac{\partial F_n}{\partial r} \delta_j \qquad \text{for } j = 2, \ldots, M-1 \tag{28}$$

The assumption that F_d is smooth over the measurement interval imposes some restriction on the measurement interval when the local gravitational field is to be recovered. The shortest wavelength of the spatial variation of the gravitational field that is significant is comparable to the altitude of the satellite. The measurement should therefore be made over a distance of approximately one-half the altitude or less. A shorter measurement interval reduces the error in performing the finite differencing for the derivative. But the measurement interval should not be made too small, otherwise the error of F_d increases again. This is due to another error source, namely, the error in the estimate of the gravitational parameter δ. When Δt decreases, the uncertainty of δ increases as $1/\sqrt{\Delta t}$ and the multiplier $1/\Delta t^2$ increases quadratically. The optimal measurement interval will depend on the measurement accuracy, data types, data volume, and geometry, among other things.

For Topex, the altitude is 1336 km, and the satellite travels a distance of about 430 km/min. The measurement interval should therefore be 1.5 min or less. Over a 10-day period, Topex ground tracks cover a zone of spherical surface between $\pm 66°$ latitudes, with distance between successive tracks less than 2.5°. With the technique described earlier, a tight grid of local gravitational field at Topex altitude can be established.

The gravitational field recovered with this method is the local gravitational field. If a global gravitational model in the form of spherical harmonic coefficients is desired, the coefficients can be computed by simple integration over the spherical surface if the satellite orbit is nearly circular, as it is in the case with Topex/Poseidon, which has an eccentricity of less than 0.001. The integration scheme is applicable due to the orthogonality property of the spherical harmonic functions. The mathematical formulation for the integration is similar to that for the Fourier transformation and is given in Appendix B. This method involves less computation but does not give precisely the same result as the conventional method.

Some limitation on the validity of the harmonic coefficients should be noted. For a satellite that does not have a polar orbit, the gravitational field around the poles is not recovered.

Since the computed spherical harmonic coefficients represent the global field, the local gravitational field around the poles will have to be provided from other sources if that part of the gravitational field is to be accurate also. In addition, the high degree harmonic coefficients cannot be recovered accurately if the satellite altitude is relatively high. A satellite to probe the high degree harmonic coefficients must fly at a low altitude. Finally, the tracking data of satellites with different orbits are sensitive to different terms of the spherical harmonic expansion. To construct a general purpose gravitational model, it is desirable to incorporate data from multiple satellites with different orbits. This can be done by combining gravitational field recovered from one satellite with a field of other sources, using the appropriate covariance matrix. These limitations are determined by the satellite and its tracking system and not by the particular method used for processing the data.

The consistency of this method can be checked in several ways. The gravitational field F_d for the same ground track but computed from an independent data set can be compared. In addition, the gravitational field at the crossing points of different ground tracks can be compared. The horizontal component of the gravitational field can be checked against the theoretical constraint that the line integral of a gravitational field around a closed path is zero. Finally, the spherical harmonic coefficients computed from different components of the gravitational field can be compared.

Estimation of Nongravitational Forces

Here we consider forces other than the Earth's gravitational attraction acting on the satellite, such as the Sun and the Moon's gravitational forces, the gravitational forces due to ocean tides, the drag, and the solar radiation. These forces are arc specific and may be included in the nominal force model. The Sun's and the Moon's gravitational forces are known with sufficient accuracy and need not be adjusted. Some satellites are drag compensated, and the nongravitational forces have little or no effect on the satellite motion. But in many cases, it may be necessary to estimate the model parameters for the other forces. Whether a parameter needs to be estimated or not depends on factors such as the arc length, the satellite altitude, and the application.

Some consideration should be given to the problem of overparameterization when additional parameters are adjusted. With a given set of measurement data, extra parameters tend to dilute the solution and in a severe case can make the solution uncertain if some parameters cannot be "separated out" from each other. Mathematically, we see the A matrix becoming rank deficient and the information matrix (i.e., the normal matrix A^TA) singular or nearly singular. Since the gravity bin approach allows maximum degree of freedom for the orbit and the forces, overparameterization occurs when extra parameters are adjusted. Here we discuss three situations where the gravity bin technique is applied. In the first case, the technique is used for orbit determination only. In the second case, it is used to recover the global gravitational coefficients with the approach based on Eq. (15). In the third case, it is used to recover the local gravitational field by Eq. (27) that is then converted into the global field.

In the first case, a short data arc, typically one or two revolutions long, is sufficient to give good results with the GPS tracking system. Except for satellites with low altitude where drag is higher, the effect of the drag and the solar radiation is relatively small, and it is not necessary to adjust the corresponding parameters. If these forces are adjusted, the estimation problem is overparameterized, and the information matrix is singular. The resulting values of the parameters are only loosely constrained by a priori information. But the orbit that is constructed from the solution is still well defined, since the problem is similar to the one discussed in Appendix A. Since we are interested in the orbit only, the result is still satisfactory.

In the second case, the method is equivalent to the conventional method of gravitational field recovery where the data arc is long and the nongravitational forces are routinely adjusted. The expansion of gravitational field in spherical harmonic coefficients is truncated and higher degree coefficients are usually stabilized with a priori constraint in the form of Kaula's rule. Furthermore, a gravitational model derived from other sources can be used as a priori information. The degree of freedom for the estimation problem is thus greatly reduced and the overparameterization no longer exists.

The third case requires extra care in the formulation of the estimation problem. The overparameterization is avoided with additional information on the bin parameters. The gravitational field is subject to a theoretical constraint in that it is the gradient of a scalar potential. The constraint is mathematically equivalent to the statement that the curl of the gravity field is zero, or that the line integral of the field around a closed path is zero. The gravitational field is often considered a conservative field due to this property. However, there should be a qualification to the last statement. Since the Earth is rotating in the inertial frame where kinetic energy is defined, this does not imply that the total energy of the satellite is conserved if the gravitational force is the only force present. For our purpose, the one statement that is most readily applied is the one regarding the line integral. We can write

$$\oint F_d(r) \cdot \mathrm{d}s = 0 \tag{29}$$

where $F_d(r)$ is the gravitational field due to the Earth, and the line integral is around a closed path. The field and the line integral are both in Earth fixed coordinates. The closed path is possible since Topex orbit has very small eccentricity. At the point where the ground track crosses itself after about one revolution, the satellite returns to approximately the same position. (An eccentricity of 0.00044 gives an altitude variation of a few kilometers.) Implicit in the previous equation is an assumption that the nominal gravitational field is derived from a scalar potential.

The gravity field can be expressed in terms of the gravity bin parameters (an approximation is made for the second time derivative at this step). The previous equation can therefore be rewritten as a constraint on the gravity bin parameters. When drag or other nongravitational forces are estimated, this constraint can be applied to the estimation process to stabilize the solution and avoid the problem of overparameterizing the system. The easiest way to implement this is to treat the previous constraint as a high-precision measurement. Since the line integral is around a closed path, the data arc chosen for data processing should normally be long enough so that the ground track crosses itself. This constraint can support the estimation of one nongravitational force parameter per closed path.

Other than drag and radiation pressure, the tidal forces may also need to be estimated. But this one does not cause the problem of overparameterization. The reason is that the same set of parameters describes the forces over a long period of time. Within that period of time, Topex would have repeated its ground track several times. The same parameters will produce different signatures in the tracking data over different repeat orbits whereas the geopotential will produce the same signature as the ground track is repeated. This difference will allow us to "separate out" the two kinds of forces. This is not the case with drag and radiation pressure where the values of force parameters are changing so rapidly that by the time the ground track is repeated, a different set of values will need to be estimated. The tidal model is easier to handle in this sense, even though the model itself may have greater complexity.

Conclusions

The accuracy of the orbit and the gravitational field obtained by this technique has been assessed by covariance analyses for Topex with a GPS tracking system.[33,34] Reference 33 discusses the orbit and gravitational field accuracy using the method of Eq. (28). Reference 34 discusses the accuracy of the gravitational field recovered with the method of Eq. (15). Theoretical consideration and the analysis of Ref. 31 shows that the orbit accuracy should be at least as good as the non-dynamic tracking. The orbit accuracy is generally in the sub-decimeter range and improves as the number of arcs is increased. The gravity field computed using Eq. (28) is accurate to a small fraction of a milligal. The gravity field recovered with Eq. (15) shows improvement in the mid-degree and order terms of the spherical harmonic coefficients over the current best gravity model. The gravity field recovered by using the method of Eq. (15) is the same as that by the conventional method based on theoretical consideration. The equivalence has been confirmed numerically with a covariance analysis that solves for a gravitational field with a small set of selected terms. Therefore, the accuracy of the gravity field is determined by the satellite and the tracking system and not by which method, the bin technique or the conventional one.

The technique presented here assumed the orbit repeats in the Earth fixed coordinates and that the tracking system provides continuous three-dimensional coverage. It employs a set of bin parameters and the satellite epoch states to solve for the satellite orbits. The Householder transformation is used to reduce the computation required to process multiple arc data.[35] The algorithm first separates the gravitational parameters from all other arc specific parameters by judicious partitioning of the measurement matrix and the Householder transformations. The transformed data from different arcs are then combined to solve for the gravitational parameters, and these results are then used to solve for the initial states of each individual arc by back substitution. The technique produces an accurate satellite orbit without relying on a high-accuracy gravitational model from other sources. An additional benefit is that the gravitational field can then be recovered, with little additional calculation.

Appendix A: Dependency of Initial State and Bin Parameters

This Appendix shows that, with the parameters we have chosen to model the satellite orbit, the adjustment term for the initial state and the bin parametrs cannot be determined completely, since the system is overparameterized. Methods to avoid this difficulty by using a priori constraint and by reducing the number of parameters are discussed. We also show that if we allow the system to be overparameterized so that the bin parameters cannot be completely determined, the satellite position and the gravitational field can still be uniquely determined from the measurements.

To show that the system is overparameterized, let us consider the regression equation for a single arc. We will assume the data type is range, but the conclusion applies to pseudorange and carrier phase also. The regression equation for the range measurement between the low Earth orbiter and the GPS satellite at time t is given by

$$\rho = \frac{\partial \rho}{\partial r_0} r_0 + \frac{\partial \rho}{\partial v_0} v_0 + \frac{\partial \rho}{\partial \delta} \delta + \frac{\partial \rho}{\partial p} p + \epsilon \tag{A1}$$

where r_0 and v_0 are adjustments to the satellite state at the epoch time and δ is the bin parameter. Terms for other parameters, including those for the clocks, the drag, and the solar radiation, are represented by p.

The partial derivatives for the initial position and velocity are given by

$$\frac{\partial \rho}{\partial r_0} = \frac{\partial \rho}{\partial r} \frac{\partial r}{\partial r_0} \tag{A2}$$

$$\frac{\partial \rho}{\partial v_0} = \frac{\partial \rho}{\partial r} \frac{\partial r}{\partial v_0} \tag{A3}$$

The partial derivative for the bin parameter is given by

$$\frac{\partial \rho}{\partial \delta} = \frac{\partial \rho}{\partial r} \frac{\partial r}{\partial \delta} = \frac{\partial \rho}{\partial r} \tag{A4}$$

To show that the system is overparameterized, we will show that if a solution exists for the regression equation, a second solution also exists. The new solution is related to the original solution by differences in the initial position and velocity so that

$$r_0' = r_0 + \Delta r \tag{A5}$$

$$v_0' = v_0 + \Delta v \tag{A6}$$

and

$$\delta' = \delta - \frac{\partial r}{\partial r_0} \Delta r - \frac{\partial r}{\partial v_0} \Delta v \tag{A7}$$

at time t. To insure that δ' remains zero at t_1 like δ, Δr and Δv are subject to the condition

$$\frac{\partial r_1}{\partial r_0} \Delta r + \frac{\partial r_1}{\partial v_0} \Delta v = 0 \tag{A8}$$

By writing a new measurement ρ' in the same form as Eq. (A1),

$$\rho' = \frac{\partial \rho}{\partial r_0} r_0' + \frac{\partial \rho}{\partial v_0} v_0' + \frac{\partial \rho}{\partial \delta} \delta' + \frac{\partial \rho}{\partial p} p + \epsilon \tag{A9}$$

and using Eqs. (A2), (A3), and (A4), we get

$$\rho' = \rho \tag{A10}$$

The substitution of Eqs. (A5), (A6), and (A7) gives the same measurement ρ for an arbitrary measurement time. If we have a set of solutions $(r_0, v_0, \delta_2, \ldots, \delta_M)$ that satisfies the regression equation, the new set of parameters $(r_0', v_0', \delta_2', \ldots, \delta_M')$ also satisfies the same equation. We thus conclude that the epoch state r_0 and v_0 and the bin parameters δ cannot be uniquely determined by the measurements alone.

In the estimation process, we may use relatively large a priori sigmas to bound the epoch state and the bin parameters. This will avoid the indefiniteness and give us a solution. However, the adjustment term to the initial state and the bin parameters are only loosely bound by the a priori sigmas, and the solution will have a large sigma. The estimation of the actual initial velocity is given by

$$\dot{r}(t_0) = v_0 + \dot{\delta}(t_0) \tag{A11}$$

which will have better accuracy than indicated by the sigma of v_0, if δ is a reasonably smooth function of time over the measurement interval Δt.

Alternatively, we may choose one arc, e.g., the first arc, as a reference arc, and set the parameter v_0 for the reference arc to zero. Estimating parameters for other arcs as before, the number of parameters to be estimated is reduced by three since the v_0 term disappears from the equation for the reference arc. This way, all parameters can be solved for without relying on a priori knowledge. The form of the equation for all arcs except the reference arc stays the same.

If we use large a priori sigmas to bound r_0, v_0, and δ, and allow the ambiguity in the solution, the position and gravitational field can still be accurately determined from the measurements. To show this, we consider the equation for position and the equation from which the gravitational field is recovered.

The position (relative to the nominal value) at measurement time other than the epoch is given by

$$r = \frac{\partial r}{\partial r_0} r_0 + \frac{\partial r}{\partial v_0} v_0 + \delta + \frac{\partial \rho}{\partial p} p \tag{A12}$$

If substitution according to Eqs. (A5), (A6), and (A7) is made for

$$r' = \frac{\partial r}{\partial r_0} r_0' + \frac{\partial r}{\partial v_0} v_0' + \delta' + \frac{\partial \rho}{\partial p} p \tag{A13}$$

we have

$$r' = r \tag{A14}$$

This shows that even though the epoch state and the bin parameters are not completely determined, the position is still uniquely determined by the measurements.

We next look at the Eq. (27) from which the gravitational field is recovered:

$$F_d = \ddot{\delta} - \frac{\partial F_n}{\partial \dot{r}} \dot{\delta} - \frac{\partial F_n}{\partial r} \delta \tag{A15}$$

By defining ξ and η as

$$\xi = \frac{\partial r}{\partial r_0} \tag{A16}$$

$$\eta = \frac{\partial r}{\partial v_0} \tag{A17}$$

we can write Eq. (A7) as

$$\delta' = \delta - \xi \Delta r - \eta \Delta v \tag{A18}$$

With the previous substitution, the expression for a new gravitational field F_d' in the form of Eq. (A15) becomes

$$F_d' = \ddot{\delta}' - \frac{\partial F}{\partial \dot{r}} \dot{\delta}' - \frac{\partial F}{\partial r} \delta' = F_d - \left(\ddot{\xi} - \frac{\partial F}{\partial \dot{r}} \dot{\xi} - \frac{\partial F}{\partial r} \xi \right) \Delta r - \left(\ddot{\eta} - \frac{\partial F}{\partial \dot{r}} \dot{\eta} - \frac{\partial F}{\partial r} \eta \right) \Delta v \tag{A19}$$

The terms in the parentheses can be evaluated with the help of variational equations for ξ and η. Since ξ and η are variational partials, they also satisfy the variational equations similar to Eq. (23):

$$\ddot{\xi} = \frac{\partial F}{\partial \dot{r}} \dot{\xi} + \frac{\partial F}{\partial r} \xi \tag{A20}$$

$$\ddot{\eta} = \frac{\partial F}{\partial \dot{r}} \dot{\eta} + \frac{\partial F}{\partial r} \eta \tag{A21}$$

Substituting Eqs. (A20) and (A21) into Eq. (A19), we get

$$F_d' = F_d \tag{A22}$$

Thus the gravitational field calculated remains the same when the substitution of Eq. (A7) is made. We can conclude that even though the bin parameters are not completely determined due to overparameterization, the gravitational field is still uniquely determined since it is not changed by the substitutions.

Note that by using Eq. (3) and Eqs. (A7), (A15), (A20), and (A21) we can show that

$$F_d = \ddot{r} - \frac{\partial F_n}{\partial \dot{r}} \dot{r} - \frac{\partial F_n}{\partial r} r - F_p \tag{A23}$$

which is an alternative to Eq. (A15) for computing the local gravitational field. Equations (A16), (A17), (A20), and (A21) also show that the variational partials $\partial r / \partial r_0$ and $\partial r / \partial v_0$ are the homogeneous solutions of the differential equation (A23).

Appendix B: Transformation of Local Gravitational Field into Spherical Harmonic Coefficients

The following shows the formulation to transform the local gravitational field to spherical harmonic coefficients using the orthogonal property of the functions. The gravitational potential is expanded in spherical harmonic functions as (see Refs. 9 and 10, Ref. 32, p. 72, and Ref. 36)

$$U = -\frac{GM}{r}\left[1 + \sum_{n=2}^{n_{\max}} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n P_n^m(\cos\theta) \times (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\right] \tag{B1}$$

where

G = universal gravitational constant
M = mass of the Earth
r = geocentric satellite distance
$n_{\max}$ = upper limit for the summation (highest degree)
a_e = Earth's mean equatorial radius
θ = satellite geocentric colatitude
λ = satellite east longitude
P_n^m = associated Legendre function
C_{nm} = gravitational coefficient for the cosine function
S_{nm} = gravitational coefficient for the sine function

Note that the symbol r has been used earlier in the variational equation with a different meaning. Differentiating the potential with respect to the coordinate r, we get the radial component of the gravitational field,

$$F_r = -\frac{\partial U}{\partial r} = -\frac{GM}{r^2}\left[1 + \sum_{n=2}^{n_{\max}} \sum_{m=0}^{n} (n+1)\left(\frac{a_e}{r}\right)^n P_n^m(\cos\theta) \times (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\right] \tag{B2}$$

The spherical harmonic function, which is the product of the associated Legendre function and the trigonometric function, has the orthogonality property expressible as

$$\iint P_n^m(\cos\theta) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} P_{n'}^{m'}(\cos\theta) \begin{Bmatrix} \cos m'\lambda \\ \sin m'\lambda \end{Bmatrix} \sin\theta \, d\theta \, d\lambda = \frac{4\pi}{(2n+1)(2-\delta_{0m})} \left[\frac{(n+m)!}{(n-m)!}\right] \delta_{nn'} \delta_{mm'} \tag{B3}$$

where δ is the Kronecker delta that is equal to one when the indices are the same and zero when they are different. The integration is over the entire spherical surface. Similar integration involving the product $\cos m\lambda \sin m'\lambda$ or $\sin m\lambda \cos m'\lambda$ is zero.

By the orthogonality of the associated Legendre functions and the trigonometric functions, we can set r to the satellite Earth-centered radius R in the equation for F_r, multiply it by a spherical harmonic function, and integrate over the entire spherical surface to get

$$\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = -\frac{1}{4\pi} \frac{R^2}{GM} \left(\frac{R}{a_e}\right)^n \frac{(2n+1)}{(n+1)} \left[\frac{(n-m)!}{(n+m)!}\right] (2-\delta_{0m}) \iint F_r(\theta,\lambda) P_n^m(\cos\theta) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \sin\theta \, d\theta \, d\lambda \tag{B4}$$

A similar equation involving the east component of the field can be derived by differentiation of the potential with respect to the coordinate λ and application of the orthogonality property,

$$\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \frac{1}{4\pi}\frac{R^2}{GM}\left(\frac{R}{a_e}\right)^n \frac{(2n+1)}{m}\left[\frac{(n-m)!}{(n+m)!}\right](2-\delta_{0m})$$
$$\iint F_\lambda(\theta,\lambda)P_n^m(\cos\theta)\begin{Bmatrix} -\sin m\lambda \\ \cos m\lambda \end{Bmatrix}\sin^2\theta \, \mathrm{d}\theta \, \mathrm{d}\lambda \tag{B5}$$

A third equation can be derived using the south component of the field. But the integral involves the east component of the field too. The equation is most conveniently derived by writing the potential as the real part of a complex potential and introducing the lowering operator of the spherical harmonics (see Ref. 36, pp. 185–188). The derivation is somewhat lengthy and is omitted here. The result is

$$\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \frac{1}{4\pi}\frac{R^2}{GM}\left(\frac{R}{a_e}\right)^n (2n+1)\left[\frac{(n-m)!}{(n+m)!}\right](2-\delta_{0m})$$
$$\iint P_n^m(\cos\theta)\left[(F_{\theta,\mathrm{even}}+\cos\theta F_{\lambda,\mathrm{even}})\begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix}\right.$$
$$\left. + (F_{\theta,\mathrm{odd}}-\cos\theta F_{\lambda,\mathrm{odd}})\begin{Bmatrix} \sin m\lambda \\ -\cos m\lambda \end{Bmatrix}\right]\sin\theta \, \mathrm{d}\theta \, \mathrm{d}\lambda \tag{B6}$$

where the even and the odd functions are defined as

$$F_{\theta,\mathrm{even}} = \frac{1}{2}\left[F_\theta(\theta,\lambda)+F_\theta(\theta,-\lambda)\right] \tag{B7}$$

$$F_{\theta,\mathrm{odd}} = \frac{1}{2}\left[F_\theta(\theta,\lambda)-F_\theta(\theta,-\lambda)\right] \tag{B8}$$

$$F_{\lambda,\mathrm{even}} = \frac{1}{2}\left[F_\lambda(\theta,\lambda)+F_\lambda(\theta,-\lambda)\right] \tag{B9}$$

$$F_{\lambda,\mathrm{odd}} = \frac{1}{2}\left[F_\lambda(\theta,\lambda)-F_\lambda(\theta,-\lambda)\right] \tag{B10}$$

These are the desired equations to convert the local gravitational field to the coefficients for the spherical harmonic functions. For numerical calculation, the integrals in Eqs. (B5) and (B6) are replaced by summations over finite terms. One way to do this is to consider all of the points on the spherical surface where the gravitational fields are available and divide the spherical surface into area elements with each point occupying one element. The integral is then approximated by a summation of the areas multiplied by the integrand evaluated at the corresponding points.

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References

[1]Wu, S. C., and Ondrasik, V. J., "Orbit Determination of Low-Altitude Earth Satellite Using GPS RF Doppler," *Proceedings of the IEEE PLANS,* Dec. 1982, pp. 85–91.

[2]Licthen, S. M., Wu, S. C., Wu, J. T., and Yunck, T. P., "Precise Positioning Capabilities for TOPEX Using Differential GPS Techniques," AAS Paper 85-401, AAS Astrodynamics Conference, Vail, CO, Aug. 12–15, 1985.

[3]Rosborough, G., "Orbit Error Due to Gravity Model Error," AAS Paper 87-534, AIAA/AAS Astrodynamics Conference, Kalispell, MT, Aug. 10–13, 1987.

[4]Melbourne, W. G., Yunck, T. P., and Wu, S. C., "GPS-Based Precision Positioning of Earth Orbiting Remote Sensing System," AAS Paper 86-398, AAS 33rd Annual Meeting—Aerospace: Century XXI, Boulder, CO, Oct. 1986.

[5]Yunck, T. P., and Wu, S. C., "Non-Dynamic Decimeter Tracking of Earth Satellites Using the Global Positioning System," AIAA Paper 86-0404, AIAA 24th Aerospace Sciences Meeting, Reno, NV, Jan. 1986.

[6]Wu, S. C., Yunck, T. P., and Thornton, C. L., "Reduced-Dynamic Technique for Precise Orbit Determination of Low Earth Satellites," AAS Paper 87-410, AIAA/AAS Astrodynamics Conference, Kalispell, MT, Aug. 10–13, 1987.

[7]Yunck, T. P., Wu, S. C., and Wu, J. T., "Strategies for Sub-Decimeter Satellite Tracking with GPS," *Proceedings of the 1986 IEEE Position Location and Navigation Symposium,* Las Vegas, NV, Nov. 1986.

[8]Yunck, T. P., Wu, S. C., and Wu, J. T., "Near-Optimal Strategies for Sub-Decimeter Satellite Tracking with GPS," *Proceedings of the Second International Symposium on Spacecraft Flight Dynamics,* Darmstadt, Germany, Oct. 1986.

[9]Kaula, W. M., *Theory of Satellite Goedesy,* Blaisdell, Waltham, MA, 1966.

[10]Heiskanen, W. A., and Moritz, H., *Physical Geodesy,* Freeman, San Francisco, CA, 1967.

[11]Colombo, O. L., "High Resolution Analysis of Satellite Gradiometry," Chapman Conference on Progress in the Determination of the Earth's Gravity Field, American Geophysical Union, Sept. 1988, pp. 85–91.

[12]Colombo, O. L., Netherlands Geodetic Commission, *Publications in Geodesy,* New Series, Vol. 7, No. 3, Delft, The Netherlands, 1984.

[13]Colombo, O. L., "Gravity Field Determination with GPS for Moving Vehicle," General Meeting of the International Association of Geodesy, Aug. 3–12, 1989, Edinburgh, Scotland, Symposium 102: *Global Positioning System; an Overview,* edited by Y. Bock and N. Leppard, Springer-Verlag, New York, 1990.

[14]Wu, J. T., "TOPEX Orbit Determination by Solving Gravity Parameters with Multiple Arcs," AAS Paper 85-411, AIAA/AAS Astrodynamics Conference, Vail, CO, Aug. 12–15, 1985.

[15]Wu, J. T., and Wu, S. C., "TOPEX Orbit Determination by Combining GPS Data from Repeat Orbits," AIAA Paper 86-2216, AIAA/AAS Astrodynamics Conference, Williamsburg, VA, Aug. 18–20, 1986.

[16]Born, G. H., Lame, D. B., and Mitchell, J. M., "A Survey of Oceanographic Satellite Altimetric Missions," *Marine Geodesy,* Vol. 8, April 1984, pp. 3–16.

[17]Born, G. H., Wunsch, C., and Yamarone, C. A., "TOPEX: Observing the Oceans from Space," *EOS Transactions, American Geophysical Union,* Vol. 65, No. 28, 1984, pp. 433–437.

[18]Butler, D. M., et al., "Earth Observing System," NASA TM-86129, Science and Mission Requirements Working Group Rept., Vol. 1, Aug. 1984.

[19]Holdaway, R., "Assessing Orbit Determination Requirements for ERS-1," AIAA Paper 86-0403, AIAA 24th Aerospace Sciences Meeting, Reno, NV, Jan. 5–9, 1984.

[20]Benz, R., Faulks, H., Langemann, M., Mecke, G., and Hieber, S., "Aristoteles—An Earth Gravity Field Recovery Mission," IAF Paper 88-116, 39th Congress of the International Astronautical Federation, Bangalore, India, Oct. 8–15, 1988.

[21]Farless, D. L., "The Application of Periodic Orbits to TOPEX Mission Design," AAS Paper 85-301, AIAA/AAS Astrodynamics Conference, Vail, CO, Aug. 12–15, 1985.

[22]Tapley, B. D., and Ries, J. C., "Orbit Determination Requirements for Topex," AAS Paper 87-429, AIAA/AAS Astrodynamics Conference, Kalispell, MT, Aug. 10–13, 1987.

[23]Marsh, J. G., et al., "A New Gravitational Model for the Earth from Satellite Tracking Data: GEM-T1," *Journal of Geophysical Research,* Vol. 93, No. B6, 1988, pp. 6169–6215.

[24]Learch, F. J., Klosko, S. M., Wagner, C. A., and Patel, G. B., "On the Accuracy of Recent Goddard Gravity Models," *Journal of Geophysical Research,* Vol. 90, No. B11, 1985, pp. 9312–9334.

[25]Tapley, B. D., Schutz, B. E., Shum, C. K., Ries, J. C., and Yuan, D. N., "An Improved Model for the Earth's Gravity Field," *Proceedings of the International Association of Geodesy Symposia,* International Union of Geodesy and Geophysics XIX General Assembly, Vancouver, Canada, Aug. 1987, pp. 125–143.

[26]Bhat, R. S., Frauenholz, R. B., and Cannell, P. E., "Topex/Poseidon Orbit Maintenance Maneuver Design," AAS Paper 89-408, AAS/AIAA Astrodynamics Conference, Stowe, VT, Aug. 7–10, 1989.

[27]Bierman, G. J., *Factorization Methods for Discrete Sequential Estimation,* Academic Press, New York, 1977.

[28]Wu, J. T., "Elimination of Clock Errors in a GPS Based Tracking

System," AIAA Paper 84-2052, AIAA/AAS Astrodynamics Conference, Seattle, WA, Aug. 20–22, 1984.

[29]Ellis, J., "Large Scale State Estimation Algorithms for DSN Tracking Station Location Determination," *Journal of the Astronautical Sciences,* Vol. 28, No. 1, 1980, pp. 15–30.

[30]Wu, J. T., Bertiger, W. I., and Wu, S. C., "Converting Gravity Bins to Spherical Harmonic Coefficients," AIAA Paper 90-2943, AAS/AIAA Astrodynamics Conference, Portland, OR, Aug. 20–22, 1990.

[31]Bertiger, W. I., Wu, J. T., and Wu, S. C., "Efficient Gravity Recovery with GPS: Operation Counts," *EOS Transactions, American Geophysical Union,* Vol. 70, No. 43, Oct. 24, 1989, p. 1059.

[32]Hatch, W., and Goad, C., "Mathematical Description of the ORAN Error Analysis Program," Planetary Science Dept. Rept. 009-73, NASA Goddard Space Flight Center, Aug. 1973.

[33]Wu, J. T., and Yunck, T. P., "Topex Orbit Determination and Gravity Recovery Using GPS Data from Repeat Orbits," AIAA Paper 88-0575, AIAA 26th Aerospace Sciences Meeting, Reno, NV, Jan. 11–14, 1988.

[34]Bertiger, W. I., Wu, J. T., and Wu, S. C., "Gravity Field Improvement Using GPS Data from Topex/Poseidon: A Covariance Analysis," AIAA Paper 90-2944, AAS/AIAA Astrodynamics Conference, Portland, OR, Aug. 20–22, 1990.

[35]Wu, S. C., Wu, J. T., and Bertiger, W. I., "An Efficient Technique for Gravity Recovery Using A Low Earth Satellite," *Proceedings of the Fifth International Geodetic Symposium on Satellite Positioning,* Las Cruces, NM, March 13–17, 1989, pp. 956–965.

[36]Merzbacher, E., *Quantum Mechanics,* Wiley, New York, 1970.

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If you have any questions or need further information on disk conversion, please telephone Richard Gaskin, AIAA Production Manager, at (202) 646-7496.

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